Ertel's Potential Vorticity and Irreversible Processes

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According to Ertel's potential vorticity the thermodynamical parameters in the equation of state define the forces which move the vortices and modify them in space and time.

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Pichler [1] raised the question if Ertel's potential vorticity theorem contains new physical points of view in addition to the "general mathematical summarisation". He referred to irreversible processes in connection with vortex development and disintegration. In the following we refer in this connection to a non-viscous medium.

Ertel's potential vorticity theorem [2] is a mathematical consequence of the hydrodynamics of ideal fluids and gases. It contains in this form both Euler's and Lagrange's formulation of the equation of motion and is valid in all systems of co-ordinates (not only in inertial ones). Ertel's theorem connects Helmholtz' and Boltzmann's vorticity theorems, as well as the circulation theorems of Kelvin and Bjerknes as special cases (see also [3–9]).

Ertel's summarised vorticity theorem can be written in the form:

$$\frac{\mathrm{D}}{\mathrm{D}t}\,[\omega_{ik}\,\mathrm{d}x^i \wedge \mathrm{d}x^k] = 2\,\partial F_i\,[\mathrm{d}x^i \wedge \mathrm{d}x^k] \tag{1}$$

where $\mathrm{D}/\mathrm{D}t$ means the absolute (covariant) and total (materialistic) derivative after time, t and $\mathrm{d}x^l \wedge \mathrm{d}x^h = -\mathrm{d}x^h \wedge \mathrm{d}x^k$ is a Cartesian differential of the second degree. $\overline{\omega}_{ik} = \partial_k u^i - \partial_i u^k$, with the velocity field

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}\,.$$

$$F_i = \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{1}{\rho}\,\partial_i\,p$$

is the acceleration field, ρ the mass density and p the pressure.

For a barotropic fluid is $p = p\,(\rho)$ or $p = \text{const}$ in the case of incompressibility. Than is $\partial_i F_h - \partial_h F_k = 0$ and equation (1) yields then (in an inertial system)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}x^k = \partial_k\,u^i\,\mathrm{d}x^k$$

Helmholtz' vorticity theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\omega_{ih} = \omega_i^l\,J_l\,u_k - \omega_k^l\,\partial_l\,u_i\,. \tag{2}$$

The conservation of vorticity is generally valid for a barotropic equation of state $p = p\,(\rho)$. If the velocity field u_i has a potential in a moment of time, then the velocity field is in all times a potential field

$$\omega_k = \partial_k\,\Phi\,.$$

For any equation of state wher p depends in addition to ρ also on other thermodynamic quantities, especially on temperature T, $p = p\,(\rho, T)$, the vorticity is not conserved. Ertel's theorem gives in this case

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\,[\omega_{ik}\,\mathrm{d}x^i \wedge \mathrm{d}x^k] &= 2\int F_k^o\,\mathrm{d}x^k \\ &= 2\int \frac{\mathrm{d}p}{\mathrm{d}T_k}\,\partial_k\left[\frac{1}{\rho}\right]\partial_i\,T\mathrm{d}x^i \wedge \mathrm{d}x^k\end{aligned} \tag{3}$$

with

$$\partial_i\,F_k - J_k\,F_i = \left[\partial_k^i\left[\frac{1}{\rho}\right]\partial_k\,T - \partial_k\left[\frac{1}{\rho}\right]\partial_i\,T\right].$$

Thus it is Ertel's theorem which allows us to introduce thermodynamic variables into the vorticity theorem. Equations (3) and (4) are the vortex dynamic form of the "zeroth main theorem" of thermodynamics. If the velocity potential Φ does exist, thermodynamic quantities cannot be defined in Ertel's representation, and the zeroth main theorem of thermodynamics is only in the case of the non-conservation of vorticity compatible with hydrodynamics.

$$\frac{\mathrm{D}}{\mathrm{D}t}\,[\omega_{ik}\,\mathrm{d}x^i \wedge \mathrm{d}x^k] = 2\,\frac{\mathrm{d}p}{\mathrm{d}T}\,\partial_i\left[\frac{1}{\rho}\right]\partial_k\,T^n\,[\mathrm{d}x^i \wedge \mathrm{d}x^k] \tag{4}$$

defines according to Ertel's theorem the temperature gradient $\partial_k\,T$.

These consequences of Ertel's vorticity theorem have great importance for dynamic meteorology. Development, motion and annihilation of atmospheric vortices

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are based on the existence of temperature gradient $\partial_i T = \text{grad } T$ in the atmosphere.

The founder of vortex dynamics, Hermann Helmholtz [10] remarked in his classical lecture on "Whirlwinds and thunderstorms" that the heat balance of the atmosphere and rotation of the Earth determine the development, form, and annihilation of cyclones. The balance of cyclones is therefore the transformation of energy with low entropy and high temperature into energy with higher entropy and lower temperature, whereas a part of this energy is provisionally stored as the kinetic energy of the vortex.

Ertel's vortex theory includes the rotation of the Earth by the introduction of the "absolute vorticity". Thermodynamic forces which produce the vortex are expressed by the right side of [4]. It follows generally that an increase of the average temperature of the atmosphere results in an increase of the number and intensity of cyclones. Concerning the application of the potential vorticity in the synoptic practice, we refer to Kleinschmidt [11], Hoskins et al. [12], and more general Pedlosky [13].

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